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On sufficient conditions for starlikeness

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Abstract. In this paper, it is shown that if $1+\Re\left(\frac{zf''(z)}{f'(z)}\right)$ takes any negative value but does not take any pure imaginary value whose modulus is larger than $\sqrt{3}$, then $f(z)$ is possible to be starlike in the open unit disk E. Another view point of result given earlier by Pfaltgraff, Reade and Umezawa (1976) is also discussed.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which are analytic in the open unit disk $E=\{z:|z|<1\}$.

Let $S^*(\alpha)$ $(0\leq\alpha<1)$ be the class of functions $f(z)$ which satisfy the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ (z\in E).$$

The function $f(z) \in \mathcal{S}^*(\alpha)$ for $0 \leq \alpha < 1$ is said to be starlike of order α in E and then $f(z)$ is univalent in E.

Let $\mathcal{C}(\alpha)$ $(0 \leq \alpha < 1)$ be the class of functions $f(z)$ which satisfy the condition

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in E).$$

The function $f(z) \in \mathcal{C}(\alpha)$ for $0 \leq \alpha < 1$ is said to be convex of order α in E and then $f(z)$ is univalent in E.

Only for the case $\alpha = 0$, if $f(z) \in \mathcal{S}^*(0)$ or $f(z) \in \mathcal{C}(0)$, we call $f(z)$ is satrlike and convex in E, respectively.

Marx [4] and Strohhäcker [8] have shown that if $f(z) \in \mathcal{C}(0)$ then $f(z) \in \mathcal{S}^*(\frac{1}{2})$. Jack [2] posed the following problem: What is the largest number $\beta = \beta(\alpha)$ such that $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\beta(\alpha))$? This problem was solved by MacGregor [3] and Wilken and Feng [9]. They proved that largest number $\beta(\alpha)$ is

$$\beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})}, & if\, \alpha \neq \frac{1}{2} \\ \frac{1}{2\log 2}, & if\, \alpha = \frac{1}{2}. \end{cases} \tag{1}$$

On the other hand, Pfaltgraff, Reade and Umezawa [7] have proved the following result contained in

Theorem A. *For each α in the interval $-\frac{1}{2} \leq \alpha < 0$, the function*

$$f_\alpha(z) = \left\{(1-z)^{2\alpha-1} - 1\right\}/(1-2\alpha)$$

satisfies

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) \geq \alpha \quad \text{in } E$$

but $f_\alpha(z)$ is not starlike in E.

Remark 1. Theorem A has not been known widely among mathematician working in the field of Complex Function Theory.

Remark 2. Theorem A shows that for arbitrary positive real number δ, if $f(z) \in \mathcal{A}$ satisfies the condition $f(z) \in \mathcal{C}(-\delta)$ or

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) > -\delta, \quad (z \in E),$$

then $f(z)$ is not necessarily starlike in E or $f(z) \notin \mathcal{S}^*(0)$.

Further if $f(z)$ and $g(z)$ are analytic in E, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ which by definition is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in E$, such that $f(z) = g(w(z))$, $z \in E$. Furthermore, if $g(z)$ is univalent in E, then we have the following equivalence (cf. e.g. [1], [5]) :

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

One of the purpose of the present paper is to give another view point of Theorem A which shows as Theorem A is natural.

2. Main Results

In this paper, we need the following Nunokawa's Lemma [6] :

Lemma. *Let $p(z)$ be analytic in E, $p(0) = 1$ and suppose that there exists a point $z_0 \in E$ such that*

$$\Re p(z) > 0 \quad \textit{for } |z| < |z_0|$$

and

$$\Re p(z_0) = 0, \quad p(z_0) = ia \quad \textit{and} \quad a \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad if\ a > 0$$

and

$$k \leq \frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad if\ a < 0.$$

Theorem B. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in E and suppose that*

$$(2) \qquad p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1 - 4z + z^2}{1 - z^2} \quad (z \in E)$$

Then $p(z)$ is a Carthéodory function or $\Re\{p(z)\} > 0$ in E.

Proof. From the hypothesis (2), it is clear that $p(z) \neq 0$ in E.

If there exists a point $z_0 \in E$, $|z_0| < 1$ such that

$$\Re\{p(z)\} > 0 \quad for\ |z| < |z_0|$$

and

$$\Re\{p(z_0)\} = 0 \quad p(z_0) = ia,\ and\ a \neq 0,$$

then from Lemma , we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik.$$

Now for the case $p(z_0) = ia$ and $a > 0$, we have

$$p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} = ia + ik = i\Im\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)$$

and

$$\Im\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) \geq \frac{1}{2}\left(3a + \frac{1}{a}\right) \geq \sqrt{3}.$$

Theorem B was obtained by Nunokawa, M, Owa, S, Takahashi, N and Saitoh, H, Sufficient conditions for Caratheodory functions, Indian J. pure. Appl. Math. 33(9), (2002), 1385-1390.

This is a contradiction of (2).

Again for the case $p(z_0) = ia$ and $a < 0$, applying the same method as above, we have

$$p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} = i\Im\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)$$

and

$$\Im\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) \geq \frac{1}{2}\left(3a + \frac{1}{a}\right) \leq -\sqrt{3}.$$

This is also a contradiction and it complete the proof.

Applying the same method, we have the following corollaries.

Corollary 1. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in E and suppose that*

$$\frac{zp'(z)}{p(z)} \prec \frac{2z}{1 - z^2} \quad (z \in E).$$

Then $p(z)$ is a Carthédory function or $\Re\{p(z)\} > 0$ in E.

Corollary 2. Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in E and suppose that

$$\frac{zp'(z)}{p(z) - \alpha} \prec \frac{2z}{1 - z^2} \quad (z \in E,\ 0 \leq \alpha < 1).$$

Then we have $\Re\{p(z)\} > \alpha$ in E.

Corollary 3. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in E and suppose that*

$$p(z) + \frac{zp'(z)}{p(z) - \alpha} \prec (1 - \alpha)\frac{1 - 4z + z^2}{1 - z^2} + \alpha \quad (z \in E,\ 0 \leq \alpha < 1).$$

Then we have $\Re\{p(z)\} > \alpha$ in E.

Corollary 4.*Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in E and suppose that*

$$\frac{zp'(z)}{p(z) - \beta} \prec \frac{2z}{1 - z^2} \quad (z \in E,\ \beta > 1).$$

Then we have $\Re\{p(z)\} < \beta$ in E.

Corollary 5.*Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E, $f(z) \neq 0$ in $0 < |z| < 1$ and* *suppose that*

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1-4z+z^2}{1-z^2} \quad (z \in E).$$

Then $f(z)$ is starlike in E or

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in E).$$

Proof. Putting

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1,$$

then it follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

From Theorem B, we obtain Corollary 5.

Remark 3. The image domain of E under the mapping

$$w = G(z) = \frac{1-4z+z^2}{1-z^2}$$

is the domain $D = \{z : |z| < \infty \textit{ and } z \neq ix,\ x \in \mathbb{R} \textit{ and } |x| \geq \sqrt{3}\}$.

From Corollary 5, we have the following result.

Corollary 5'. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and for arbitrary positive real number δ for which $f(z)$ satisfies*

(3) $$1 + \Re\frac{zf''(z)}{f'(z)} > -\delta \quad (z \in E).$$

Then $f(z)$ is not necessarily satrlike in E.

Proof. If $f(z)$ satisfies the condition (3) but $1+\frac{zf''(z)}{f'(z)}$ takes purely imaginary value whose modulus is larger than $\sqrt{3}$, then $f(z)$ is not necessarily starlike in E.

Remark 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and let us put

$$\frac{zf'(z)}{f(z)} = \frac{1-z}{1+z},$$

then it is easy to see that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 - 4z + z^2}{1 - z^2}$$

$$\Rightarrow 1 + \frac{e^{i\theta} f''(e^{i\theta})}{f'(e^{i\theta})} = i \left(\frac{\cos\theta - 2}{\sin\theta} \right).$$

Here

$$\left| \frac{\cos\theta - 2}{\sin\theta} \right| \geq \sqrt{3} \quad for \quad 0 \leq \theta \leq 2\pi,$$

and

$$\lim_{|z| \to 1^-} \left(1 + \Re \frac{zf''(z)}{f'(z)} \right) = -\infty.$$

This shows that if $1 + \Re\frac{zf''(z)}{f'(z)}$ *takes any negative real value but does not takes any pure imaginary value whose modulus is larger than* $\sqrt{3}$, *then* $f(z)$ *is possible to be starlike in* E.

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